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A Cellular Automaton Model for Two-Lane Traffic

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We present some long time limit properties of a cellular automaton that models traffic of cars on a (infinite) two-lane road. This model, called TL184, is a natural generalization of the cellular automaton classified as 184 by Wolfram (to be abbreviated by CA184) and studied before as a model for one-lane traffic. TL184 models cars' motions on each lane by particles that interact via the CA184 rules, and cars' lane changes by a possibility for particles to flip from one CA184 to another. We calculate the infinite-time limit of the particle current in TL184, starting from a translation invariant measure, and use this result to show how the possibility of lane changes may enhance the current of cars in TL184 compared to that in a corresponding model of two non-interacting one-lane roads. We provide examples which demonstrate that even though the rules that regulate lane changes are completely symmetric, the system does not evolve to an equipartition of cars among both lanes from a given initially asymmetric distribution; moreover, the asymptotic car velocities and currents may be different on different lanes. We also show that, for a particular class of initial distributions, the asymptotic car density on a lane may be a non-monotonic function of the initial car density on this lane. Finally, we derive the current-density relation for an extended continuous-time version of TL184 with asymmetric lane-changing rules.

KEY WORDS: Cellular automata; two-lane traffic; current-density relation.

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1. INTRODUCTION

Traffic flow phenomena have attracted considerable interest in recent years (see, for instance [Wo], [SW], [CSS] and [HH] and references therein), both from the applied and theoretical points of view. The main goal in a mathematical study of traffic flow is usually to understand how macroscopic phenomena, like the current-density relation (the flow diagram in traffic engineering language) or traffic jams, emerge from the microscopic laws governing the local interaction among the individual components, i.e., the cars. In this sense, the elementary cellular automaton that has the number 184 in Wolfram's classification ([W]) is one of the most simple models of single-lane traffic, exhibiting basic empiric properties of traffic flow such as a single maximum in the current-density relation or the occurrence of stable shocks (traffic jams) in the presence of obstacles ([BML], [TE], [NH], [DE], [ERS]). In the present paper, we study a two-lane traffic model constructed on the basis of this cellular automaton. The two lanes—which are each represented by a cellular automaton 184—interact through the possibility of lane changes. We show that this model exhibits some non-trivial and quite interesting properties. Due to its simplicity, this model is amenable to rigorous analytical analysis, which is usually more difficult in other two-lane traffic models suggested in the literature that try to identify the simplest "realistic" models (see [CWS], [NWWS], [NWW], [Na1], [Na2], [RNSL]).

The cellular automaton 184 (abbreviated by CA184) models a one-lane road by $\mathbf{Z}$, and cars on this road by identical particles. These particles occupy the sites of $\mathbf{Z}$ under the constraint that there may be at most one particle per site. Their discrete-time update is synchronous (all particles try to move simultaneously) with the evolution rule that at each time step each particle tries to jump one position to the right, but succeeds only if that position was empty. Denote a configuration of particles in CA184 at time $n \in \mathbf{N}$ by $\zeta_n \in \{0, 1\}^{\mathbf{Z}}$ with $\zeta_n(i)=1$ ($\zeta_n(i)=0$) indicating the presence (absence) of a particle at the site i at time n. The formal definition of the particle dynamics in CA184 may be given by the equation:

$$\zeta_{n+1}(i)=\zeta_n(i-1)(1-\zeta_n(i))+\zeta_n(i)\,\zeta_n(i+1), \qquad \text{for all} \quad i \in \mathbf{Z}, \text{ and all } n \in \mathbf{N}.$$

The two-lane traffic model constructed and studied here will be called *Two-Lane 184* and abbreviated by TL184. In this model, the two lane road is represented by the collection of sites $G \equiv \{(i, j), i \in \mathbf{Z}, j \in \{1, 2\}\}$. We call $G_1 \equiv \{(i, 1), i \in \mathbf{Z}\}$ *lane 1* or *the upper lane* and $G_2 \equiv \{(i, 2), i \in \mathbf{Z}\}$ *lane 2* or *the lower lane*. We say that site (i, j) corresponds to *position* i on lane j. In TL184, each site of G can be occupied by at most one particle, corresponding to the presence of a car at that site on the "road". Let us denote a configuration of TL184 at time $n \in \mathbf{N}$ by $\eta_n \in \mathscr{G} := \{0, 1\}^G$ with $\eta_n(i, j) = 1$ ($\eta_n(i, j) = 0$) indicating the presence (absence) of a particle at position i of lane j (i.e., the site (i, j)) at time n. The time evolution of the particles generalizes that of CA184 as follows: If a site (i, j) is occupied by a particle at time $n \in \mathbf{N}$ then at time $n+1$, this particle may be either at $(i+1, j)$, or at $(i+1, j')$, $j' \neq j$, or at (i, j). The choice among these three possibilities is determined as follows. First the particle tries to jump to position $(i+1, j)$ (as in CA184) and succeeds if and only if that position was vacant, that is, if $\eta_n(i+1, j) = 0$. If this jump is forbidden, it tries to change lanes and to occupy the site $(i+1, j')$ where $j' \neq j$. The change of lane will occur provided that the target site is vacant and there is no particle at site (i, j') about to jump to it, i.e., the lane change occurs if and only if $\eta_n(i+1, j) = 1$ and $\eta_n(i+1, j') = \eta_n(i, j') = 0$. Finally, if both moves (ahead and lane-change) are not allowed, the particle stays where it is. Denoting by $\bullet$ ($\circ$) an occupied (empty) site, a lane change from the upper to the lower lane may be illustrated by the following diagram

$$\begin{pmatrix} \bullet & \bullet \\ \circ & \circ \end{pmatrix} \rightarrow \begin{pmatrix} \circ & \bigstar \\ \bigstar & \bullet \end{pmatrix} \tag{1}$$

where the $\bigstar$'s can be either $\circ$ or $\bullet$ depending on the environment; a lane change from the lower to the upper lane may be similarly illustrated. A complete graphical presentation of the dynamical rules is given below in the column "TL184" of Table 1. A more formal definition of TL184 may be given by an operator $T: \{0, 1\}^G \rightarrow \{0, 1\}^G$ such that $\eta_{n+1} := T(\eta_n)$. The explicit expression of T is given in Section 2, Eq. (18), where it is needed. Notice that for any time $n \in \mathbf{N}$ and any $i \in \mathbf{Z}$, $\eta_{n+1}(i, 1)$ and $\eta_{n+1}(i, 2)$ depend solely on $\{\eta_n(k, \ell), k = i-1, i, i+1, \ell = 1, 2\}$.

TL184 has been constructed as a system consisting of two CA184 that interact through the possibility of lane changes. For the purpose of comparison, let us denote by DCA184 the system of two non-interacting CA184 where each particle can only move (or try to) on its lane and does so according to the CA184 evolution rules. Formally, DCA184 is a discrete-time process η_n, $n \in \mathbf{N}$, with the state space $\{0, 1\}^G$ whose evolution is given by $\eta_{n+1}(i, j) = \eta_n(i-1, j)(1 - \eta_n(i, j)) + \eta_n(i, j)\, \eta_n(i+1, j)$, for all $i \in \mathbf{Z}$, $n \in \mathbf{N}$, $j = 1, 2$.

The two models considered here, DCA184 and TL184, exhibit a particle-vacancy symmetry inherited from CA184. Informally this symmetry, that will be used many times below, means that both models can be defined by setting up either the rules that describe how particles move to the right or the rules that describe how vacancies move to the left, and those rules

Table 1. The Column Entitled "3-State" Presents the Dynamics (16) of the 3-State Cellular Automaton. It Has the Pattern $(\xi(i-1), \xi(i), \xi(i+1)) \to S(\xi)(i)$. The Column Entitled "TL184" Presents the Dynamics of TL184 in the Form $(\eta_n(i-1), \eta_n(i), \eta_n(i+1)) \to \eta_{n+1}(i)$. Configurations of TL184 Which Are Equivalent up to an Interchange of Lanes Are Not Listed Separately. The Configurations of the 3-State Cellular Automaton and Those of TL184 Are Matched to Simplify Checking (17). Namely, Each 3-State Configuration Is the Result of the Application of M to the TL184 State in the Same Line of the Table

3-state	TL184	3-state	TL184	
$(000)\to 0$	$\begin{matrix}\circ&\circ&\circ\\\circ&\circ&\circ\end{matrix}\to\begin{matrix}\circ\\\circ\end{matrix}$	$(111)\to 1$	$\begin{matrix}\bullet&\bullet&\bullet\\\circ&\circ&\circ\end{matrix}\to\begin{matrix}\circ\\\bullet\end{matrix}$	$\begin{matrix}\bullet&\bullet&\circ\\\circ&\circ&\bullet\end{matrix}\to\begin{matrix}\circ\\\bullet\end{matrix}$
$(001)\to 0$	$\begin{matrix}\circ&\circ&\bullet\\\circ&\circ&\circ\end{matrix}\to\begin{matrix}\circ\\\circ\end{matrix}$		$\begin{matrix}\circ&\bullet&\bullet\\\bullet&\circ&\circ\end{matrix}\to\begin{matrix}\circ\\\bullet\end{matrix}$	$\begin{matrix}\bullet&\circ&\bullet\\\circ&\bullet&\circ\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$
$(002)\to 0$	$\begin{matrix}\circ&\circ&\bullet\\\circ&\circ&\bullet\end{matrix}\to\begin{matrix}\circ\\\circ\end{matrix}$	$(011)\to 0$	$\begin{matrix}\circ&\bullet&\bullet\\\circ&\circ&\circ\end{matrix}\to\begin{matrix}\circ\\\circ\end{matrix}$	$\begin{matrix}\circ&\bullet&\circ\\\circ&\circ&\bullet\end{matrix}\to\begin{matrix}\circ\\\circ\end{matrix}$
$(010)\to 0$	$\begin{matrix}\circ&\bullet&\circ\\\circ&\circ&\circ\end{matrix}\to\begin{matrix}\circ\\\circ\end{matrix}$	$(101)\to 1$	$\begin{matrix}\bullet&\circ&\bullet\\\circ&\circ&\circ\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$	$\begin{matrix}\bullet&\circ&\circ\\\circ&\circ&\bullet\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$
$(012)\to 1$	$\begin{matrix}\circ&\bullet&\bullet\\\circ&\circ&\bullet\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$	$(110)\to 1$	$\begin{matrix}\bullet&\bullet&\circ\\\circ&\circ&\circ\end{matrix}\to\begin{matrix}\circ\\\bullet\end{matrix}$	$\begin{matrix}\bullet&\circ&\circ\\\circ&\bullet&\circ\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$
$(020)\to 0$	$\begin{matrix}\circ&\bullet&\circ\\\circ&\bullet&\circ\end{matrix}\to\begin{matrix}\circ\\\circ\end{matrix}$	$(112)\to 2$	$\begin{matrix}\bullet&\bullet&\bullet\\\circ&\circ&\bullet\end{matrix}\to\begin{matrix}\bullet\\\bullet\end{matrix}$	$\begin{matrix}\bullet&\circ&\bullet\\\circ&\bullet&\bullet\end{matrix}\to\begin{matrix}\bullet\\\bullet\end{matrix}$
$(021)\to 1$	$\begin{matrix}\circ&\bullet&\bullet\\\circ&\bullet&\circ\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$	$(121)\to 1$	$\begin{matrix}\bullet&\bullet&\bullet\\\circ&\bullet&\circ\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$	$\begin{matrix}\bullet&\bullet&\circ\\\circ&\bullet&\bullet\end{matrix}\to\begin{matrix}\circ\\\bullet\end{matrix}$
$(022)\to 2$	$\begin{matrix}\circ&\bullet&\bullet\\\circ&\bullet&\bullet\end{matrix}\to\begin{matrix}\bullet\\\bullet\end{matrix}$	$(211)\to 1$	$\begin{matrix}\bullet&\bullet&\bullet\\\bullet&\circ&\circ\end{matrix}\to\begin{matrix}\circ\\\bullet\end{matrix}$	$\begin{matrix}\bullet&\bullet&\circ\\\bullet&\circ&\bullet\end{matrix}\to\begin{matrix}\circ\\\bullet\end{matrix}$
$(100)\to 1$	$\begin{matrix}\bullet&\circ&\circ\\\circ&\circ&\circ\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$	$(202)\to 2$	$\begin{matrix}\bullet&\circ&\bullet\\\bullet&\circ&\bullet\end{matrix}\to\begin{matrix}\bullet\\\bullet\end{matrix}$	
$(102)\to 1$	$\begin{matrix}\bullet&\circ&\bullet\\\circ&\circ&\bullet\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$	$(210)\to 1$	$\begin{matrix}\bullet&\bullet&\circ\\\bullet&\circ&\circ\end{matrix}\to\begin{matrix}\circ\\\bullet\end{matrix}$	
$(120)\to 0$	$\begin{matrix}\bullet&\bullet&\circ\\\circ&\bullet&\circ\end{matrix}\to\begin{matrix}\circ\\\circ\end{matrix}$	$(212)\to 2$	$\begin{matrix}\bullet&\bullet&\bullet\\\bullet&\circ&\bullet\end{matrix}\to\begin{matrix}\bullet\\\bullet\end{matrix}$	
$(122)\to 2$	$\begin{matrix}\bullet&\bullet&\bullet\\\circ&\bullet&\bullet\end{matrix}\to\begin{matrix}\bullet\\\bullet\end{matrix}$	$(220)\to 0$	$\begin{matrix}\bullet&\bullet&\circ\\\bullet&\bullet&\circ\end{matrix}\to\begin{matrix}\circ\\\circ\end{matrix}$	
$(200)\to 2$	$\begin{matrix}\bullet&\circ&\circ\\\bullet&\circ&\circ\end{matrix}\to\begin{matrix}\bullet\\\bullet\end{matrix}$	$(221)\to 1$	$\begin{matrix}\bullet&\bullet&\bullet\\\bullet&\bullet&\circ\end{matrix}\to\begin{matrix}\bullet\\\circ\end{matrix}$	
$(201)\to 2$	$\begin{matrix}\bullet&\circ&\bullet\\\bullet&\circ&\circ\end{matrix}\to\begin{matrix}\bullet\\\bullet\end{matrix}$	$(222)\to 2$	$\begin{matrix}\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\end{matrix}\to\begin{matrix}\bullet\\\bullet\end{matrix}$	

will be essentially the same. More precisely, if R: $\{0, 1\}^G \to \{0, 1\}^G$ is the reflection operator defined by $(R(\eta))(i, j) = \eta(-i, j)$, $i \in \mathbf{Z}, j = 1, 2$, and E: $\{0, 1\}^G \to \{0, 1\}^G$ is the vacancy/particle exchange operator defined by $(E(\eta))(i, j) = 1 - \eta(i, j)$, $i \in \mathbf{Z}, j = 1, 2$, then the equivalence of the rules as stated above is a consequence of the following commutation relation

$$T(R(E(\eta))) = R(E(T(\eta))), \qquad \forall \eta \in \{0, 1\}^G \tag{2}$$

which is straightforward to verify by applying the evolution rules (18) of TL184.

Let us review our results. They concern the infinite-time-limit of certain characteristics of TL184 (the particle current, density and velocity, both in the system as a whole, and on each separate lane) which will be alternatively called "asymptotic", or "limiting", or "stationary".

In Theorem 1, we show that when TL184 starts from a translation invariant measure μ, then the infinite-time limit of the particle current is the sum of analogous limits in two non-interacting (!) CA184, starting on each lane from particular measures ν_1 and ν_2. The construction of ν_1, ν_2 from μ will be given explicitly in (6). We note that Theorem 1 allows one to express the asymptotic particle current as a function of the parameters of the initial (!) distribution of particles in TL184. This fact is stressed in Corollary 1 that states that this current is $\min\{\rho_1 + \rho_2, 2 - \rho_1 - \rho_2\}$, when TL184 starts from the Bernoulli measure with the particle density ρ_i on lane i, $i = 1, 2$. This initial measure is also studied in Corollary 2. There we show that the asymptotic current in TL184 may be larger than that in DCA184, when both start from the same measure, and when either $\rho_1 < 1/2 < \rho_2$ or $\rho_2 < 1/2 < \rho_1$. This current enhancement in TL184 is a consequence of lane changes, and thus it suggests that TL184 be a meaningful model of two-lane traffic. [Let us note that it is not difficult to construct some specifically chosen initial configurations of particles on G, such that at a given time n the current in TL184 is larger than that in DCA184. However, this fact does not diminish the significance of Corollary 2, since it applies to the case when the initial measure is "sufficiently disordered": it is Bernoulli with different particle densities on the lanes.]

At the heart of the proof of Theorem 1 there is a coupling of TL184, starting from μ, with DCA184, starting from another measure ν, in such a way that their particle currents are equal at all times (ν_1 and ν_2 mentioned above are the particle distributions on lanes 1 and 2 in the measure ν). Since the particle densities in ν_1 and in ν_2 are equal, this coupling may leave the impression that particles in TL184 manage to rearrange themselves equally on both lanes, as time increases. This, however, does not happen in general, as Theorem 2 indicates. It shows that the stationary particle density on lane 1 is different from that on lane 2, when, initially, the measure on lane 1 is Bernoulli while the other lane is entirely empty (or entirely occupied, which follows from particle-vacancy symmetry). The initial measure considered in Theorem 2 satisfies the assumptions of Corollary 1, but there is no contradiction between these statements since the fact that the limiting current depends solely on $\rho_1+\rho_2$ (as Corollary 1 asserts) does not necessarily require that $(\rho_1+\rho_2)/2$ is the limiting particle density on each lane. Theorem 2 and Remark 2 also show that the long time limit of the particle current and the long time limit of the particle velocity may be different on different lanes. That these notions actually make sense is established by our Proposition 1. It asserts that each particle of TL184, starting from a product measure, will change lanes only a finite number of times and thus, will settle sooner or later on one of the lanes.

The main ingredient in the proof of Theorem 2 is an analysis of the dispersion (as time increases) of particles that form a block on a lane in the initial state of TL184 (see Figure 2). This analysis is complete, when those sites on the other lane which are neighbors to the particles from a block, are all vacant, the condition reflected in the theorem assumptions. The control we have on the block dispersion mechanism allows us to reveal one more intriguing property of TL184: the stationary particle density on a lane may be a non-monotone function of the initial particle density on the same lane. Namely, we construct a specific measure by leaving lane 1 empty and putting particles on lane 2 with the measure ν_ρ^{184}, which is the stationary distribution of CA184, starting from the Bernoulli measure with the particle density ρ. We then show (Theorem 3) that for such initial measures the stationary particle densities on each lane of TL184 are both strictly smaller than 1/2, if the initial density ρ on the non-empty lane is different from 1/2 and 1. This result implies that the limiting particle density on lane 2 is a non-monotonic function of ρ (see Corollary 3 and Figure 3). Note that the constructed initial measure relates to the real situation when the cars are allowed to change lanes only after they have achieved a steady state on one lane (e.g. when a one-lane road widens up to two lanes). This relation and Theorem 3 and Corollary 3 motivated us to investigate whether a measure of the form $\nu_{\rho_1}^{184}\times\nu_{\rho_2}^{184}$ might be invariant for TL184; in this measure the lanes are independent and the particles on lane i are distributed by $\nu_{\rho_i}^{184}$, $i=1,2$. In Proposition 2 we show that $\nu_{\rho_1}^{184}\times\nu_{\rho_2}^{184}$ is not invariant, when $\rho_1 < 1/2 < \rho_2$. In the proof, we adapted the method that had allowed us to characterize the invariant measures of CA184 ([BF2]). We believe that a further development of this method may lead to an analogous characterization for TL184.

We remark that less restrictive conditions on the initial measures may lead to results similar to Theorems 2 and 3. How far these results can be pushed remains an open question.

The last section (Section 4) discusses some questions associated with a continuous time version of TL184. The current enhancement due to lane change observed in Section 2 is reconsidered in this version of the model. In particular, we introduce an asymmetric lane-changing rule which requires cars to return to "lane 2" immediately after passing (assuming this to be possible). This rule is motivated by traffic regulations in some countries (e.g. Germany) which aim at keeping one lane (lane 1 in our notation) free for passing by faster vehicles.

2. THE LONG TIME LIMIT OF THE PARTICLE CURRENT IN TL184

In this section we prove that the long time limit of the particle current for TL184 exists and we compute its value (Theorem 1 and Corollary 1). We then compare this limiting current to that in the process consisting of two independent single-lane models, DCA184, starting from the same distribution (Corollary 2).

Let $N^n_{(0,1)}(\eta)$ denote the number of particles that hop from position (resp., site) 0 to position (resp., site) 1 in G (resp., $\mathbf{Z}$) at time n in TL184 or in DCA184 (resp., CA184) that starts from the configuration η[4]. For a translation invariant measure μ, we call

$$J^*_{t=n}(\mu) := \mathbf{E}_\mu N^n_{(0,1)}(\eta) \tag{3}$$

the particle current at time n for the initial measure μ. Here and in (4) below, * indicates the process we are considering, and thus, may be either TL184 or DCA184 or CA184. Correspondingly

$$J^*_{t=\infty}(\mu) := \lim_{n\to\infty} J^*_{t=n}(\mu) \tag{4}$$

is called *the long time limit of the particle current (or stationary current) for the initial measure* μ. If μ is invariant we use $J^*(\mu) := J^*_{t=n}(\mu)$.

Theorem 1 will express $J^{\mathrm{TL184}}_{t=\infty}(\mu)$ with the help of an auxiliary measure ν. We now present the construction of ν from μ.

Let us say that we have an *isolated* particle in configuration η at site $(i, j) \in G$, if the corresponding site on the other lane is empty (that is,

[4] We remind the reader that by referring to a *position* in G without specification of the lane we have in mind both sites $(i, 1)$, $(i, 2)$ at position i. Hence $N^n_{(0,1)}(\eta)$ may take the values 0,1 or 2 respectively.

$\eta(i, j)=1$ and $\eta(i, j')=0$, for $j' \neq j$). Then we define $\mathscr{S}$ as the set of configurations on which all consecutive isolated particles are in different lanes (in between two consecutive isolated particles there may be positions that are empty or occupied by two particles).

With an arbitrary translation invariant measure μ on $\mathscr{G}=\{0,1\}^G$ we now associate a translation invariant measure ν which is supported by $\mathscr{S}$. Our construction of ν from μ may be informally described as follows. It does not modify non-isolated particles and non-isolated vacancies. It takes the particle of μ which is the nearest to the position 0 from the right among all isolated particles, and distributes it equally among the lanes. Given the lane occupied by this particle, all other isolated particles of μ are put by ν in the alternating order on the lanes, so that each configuration from the support of ν is from $\mathscr{S}$, as desired. Let us now present a formal construction of ν. Let $M:\{0,1\}^G \to \{0,1,2\}^{\mathbf{Z}}$ be "the particle counting operator" defined by

$$M(\eta)(i) := \eta(i,1)+\eta(i,2), \qquad \eta \in \{0,1\}^G \tag{5}$$

Given $\eta \in \mathscr{G}$, we define two configurations $L^1(\eta)$ and $L^2(\eta)$ by postulating that *(a)* both belong to $\mathscr{S}$; *(b)* their total occupation numbers coincide with those of η: $M(\eta)=M(L^1(\eta))=M(L^2(\eta))$; *(c)* the configuration $L^1(\eta)$ ($L^2(\eta)$) has the first isolated particle to the right of the origin in lane 1 (2). A formal construction of ν is provided via its definition on cylinder subsets of $\mathscr{S}$:

$$\nu(A) := \tfrac{1}{2} \sum_{i=1,2} \mu\{\eta \in \mathscr{G} : \zeta = L^i(\eta) \text{ for some } \zeta \in A\}, \qquad \forall \text{ cylinder } A \subset \mathscr{S} \tag{6}$$

To state and to prove Theorem 1, we shall need certain facts on the asymptotic particle current behavior in CA184. Let us recall from [BF2][5] what is known in this context (more details are given in Note added 2).

In [BF2], we prove that if λ is a translation invariant measure on $\{0,1\}^{\mathbf{Z}}$, then CA184, starting from λ, converges to a measure, which we denote by λ^∞, that is also translation invariant. This convergence implies, in particular, that $J^{\text{CA184}}_{t=\infty}(\lambda)$ exists since it is just $J^{\text{CA184}}(\lambda^\infty)$. An important fact, however, is that we can express its value without knowing λ^∞. Namely, let us represent λ as

$$\lambda = a\gamma + (1-a)\,\sigma \tag{7}$$

[5] Certain ideas employed in [BF2] have already appeared in the literature; see for example, [BF1], [KS], [NH], [BML].

for some $a \in [0, 1]$ and some translation invariant measures γ and σ that satisfy

$$\text{for } \gamma\text{-almost every configuration } \zeta \in \{0, 1\}^{\mathbf{Z}}, \text{ if } \zeta(0) = 1$$

$$\text{then } \sum_{k=0}^{n(\zeta)} (1 - 2\zeta(k)) = 0 \text{ for some finite positive } n(\zeta); \tag{8}$$

$$\text{for } \sigma\text{-almost every configuration } \zeta \in \{0, 1\}^{\mathbf{Z}}, \text{ if } \zeta(0) = 0$$

$$\text{then } \sum_{k=0}^{n(\zeta)} (1 - 2\zeta(k)) = 0 \text{ for some finite positive } n(\zeta) \tag{9}$$

Then,

$$J_{t=\infty}^{\mathrm{CA184}}(\lambda) = a\rho(\gamma) + (1-a)(1-\rho(\sigma)) \tag{10}$$

where $\rho(\lambda) := P_\lambda[\zeta(0) = 1]$ is the particle density of the measure λ.

Note that there always exists a unique expansion

$$\lambda = \sum_i a_i \beta_i^{>1/2} + \sum_j b_j \beta_j^{<1/2} + \sum_k c_k \beta_k^{=1/2} \tag{11}$$

where a's, b's and c's are positive reals summing up to 1, and each β is an ergodic translation invariant measure on $\{0, 1\}^{\mathbf{Z}}$ whose particle density is $> 1/2$, or $< 1/2$, or $= 1/2$, as indicated in the superscript. Because of the ergodicity, any $\beta^{>1/2}$ satisfies (8), any $\beta^{<1/2}$ satisfies (9), and any $\beta^{=1/2}$ satisfies both. Thus, the expansion (7) is always possible since it may constructed from (11); it may be not unique, since $\beta^{=1/2}$ can be incorporated in ether γ or in σ, but this non-uniqueness does not affect the value of (10), because $\rho(\beta^{=1/2}) = 1 - \rho(\beta^{=1/2})$.

We can now state the central result of the present section.

Theorem 1 (The particle current in TL184 in the long time limit). Let μ be a translation invariant measure on $\mathcal{G}$, let ν be constructed from μ via (6) and let ν_i denote the distribution of the particles on the i-th lane ($i = 1, 2$) in ν. Then $J_{t=\infty}^{\mathrm{TL184}}(\mu)$, the long time limit of the particle current in TL184, starting from μ, exists and is given by

$$J_{t=\infty}^{\mathrm{TL184}}(\mu) = J_{t=\infty}^{\mathrm{CA184}}(\nu_1) + J_{t=\infty}^{\mathrm{CA184}}(\nu_2). \tag{12}$$

Its numeric value may be found by calculating $J_{t=\infty}^{\mathrm{CA184}}(\nu_i)$, $i = 1, 2$, following (7–10).

Corollary 1 (A case when the limiting current is a function of the initial particle densities). If μ is a measure on $\mathcal{G}$ that puts particles independently on G with the density ρ_i on the i-the lane, $i=1,2$ (that is, $\rho_1=P_\mu[\eta(i,1)=1]$ and $\rho_2=P_\mu[\eta(i,2)=1]$), then

$$J^{\mathrm{TL184}}_{t=\infty}(\mu)=\min\{\rho_1+\rho_2,2-\rho_1-\rho_2\}. \tag{13}$$

Theorem 1 states that if μ is translation invariant then $J^{\mathrm{TL184}}_{t=\infty}(\mu)$ exists and its value may be calculated in two steps: firstly, one constructs ν from μ; secondly, for $i=1,2$, one finds the expansion (7) for ν_i and uses it to express $J^{\mathrm{CA184}}_{t=\infty}(\nu_i)$ via (10). Plugging $J^{\mathrm{CA184}}_{t=\infty}(\nu_i)$, $i=1,2$, in (12) gives the result. Corollary 1 presents a particular case for which both steps are dispensable. This case is of particular interest because it allows us to exhibit the following property of TL184.

Corollary 2 (Current enhancement due to lane changes in TL184). Let μ be as in Corollary 1. Then

$$\begin{aligned}J^{\mathrm{TL184}}_{t=\infty}(\mu)-J^{\mathrm{DCA184}}_{t=\infty}(\mu)&=\min\{\rho_1+\rho_2\,,\;2-\rho_1-\rho_2\}\\&\quad-\min\{\rho_1,1-\rho_1\}-\min\{\rho_2,1-\rho_2\}\begin{cases}>0 \text{ if } \rho_1<\frac{1}{2}<\rho_2 \text{ or } \rho_2<\frac{1}{2}<\rho_1\\=0 \text{ otherwise}\end{cases}\end{aligned}$$

Proof of Corollary 1. The structure of μ and the construction of ν from μ imply that ν puts particles independently on the sites of G, with the density $(\rho_1+\rho_2)/2$ on both lanes; in other words, ν_i, $i=1,2$, is Bernoulli with the particle density equal to $(\rho_1+\rho_2)/2$. Thus, if $(\rho_1+\rho_2)/2\leqslant 1/2$ then both ν_1 and ν_2 satisfy (8), while if $(\rho_1+\rho_2)/2\geqslant 1/2$ then both satisfy (9). From (7)–(8)–(9)–(10), we then have that

$$J^{\mathrm{CA184}}_{t=\infty}(\nu_1)=J^{\mathrm{CA184}}_{t=\infty}(\nu_2)=\min\{(\rho_1+\rho_2)/2\,,\,1-(\rho_1+\rho_2)/2\}, \tag{14}$$

which together with (12) imply (13). ∎

Proof of Corollary 2. Let μ_i denote the distribution of particles in the i-th lane ($i=1,2$) in μ. Employing the fact that both are Bernoulli and reasoning as in the proof of Corollary 1, we get that $J^{\mathrm{CA184}}_{t=\infty}(\mu_i)=$

$\min\{\rho_i, 1-\rho_i\}$, $i=1, 2$. Now, since, by the very definition of DCA184, the particles do not change lanes in this process then

$$\begin{aligned} J^{\mathrm{DCA184}}_{t=\infty}(\mu) &= J^{\mathrm{CA184}}_{t=\infty}(\mu_1) + J^{\mathrm{CA184}}_{t=\infty}(\mu_2) \\ &= \min\{\rho_1, 1-\rho_1\} + \min\{\rho_2, 1-\rho_2\} \end{aligned} \tag{15}$$

Combining this with (13) proves the corollary. ∎

Proof of Theorem 1. The proof goes as follows. First we show that starting from any configuration from $\mathcal{S}$, the evolution under the TL184 rules never has lane-changes and therefore is exactly the same as if the evolution on both lanes were independent, that is, as if the dynamics were that of DCA184. Next we define a map of the two-lane model into an auxiliary one-dimensional 3-state cellular automaton that preserves all the information necessary to compute currents. Finally, we construct a measure m concentrated on a subspace of $\mathcal{G} \times \mathcal{S}$ such that its first (resp., second) marginal is μ (resp., ν from (6)) and such that if (η, η') belongs to the support of m then both η and η' are mapped to the same 3-state configuration. The existence of this measure m, the preservation property of the 3-state cellular automaton and the property of $\mathcal{S}$ as stated above, ensure that the current in TL184, starting from μ, is the same (at each time) as the current in DCA184, starting from ν. The long time limit of the latter exists and is expressed by the r.h.s. of (12) due to the relation of DCA184 to CA184. This would prove the theorem.

The first step of the proof is the claim that particles of TL184 never change lanes, when starting from a configuration from the set $\mathcal{S}$ (defined in the beginning of this section). This will be proven once we have verified that $\mathcal{S}$ is closed under TL184 evolution and that, from any configuration in this set, a single time step never involves lane-changes.

To simplify the exposition, we say that each isolated particle (defined in the beginning of the section) in a given configuration receives one of two *opposite* labels: *up* or *down*, according to its position in lane 1 or 2, respectively, and that each particle in a doubly occupied position receives no label. We also say that a non-isolated particle has *a companion*, namely, the particle at the same position on the other lane. Then, $\mathcal{S} \subset \mathcal{G}$ is the set of configurations in which all consecutive isolated particles have different labels. Let η_0 be a configuration in $\mathcal{S}$ and η_1 the next configuration under the evolution of TL184. We use now Table 1 to check the following three facts concerning the associated "label dynamics" that, taken together, imply the claim about the evolution of TL184 on $\mathcal{S}$.

(1) A particle, call it A, that is isolated in η_0 and able to move will no longer be isolated in η_1 only if in η_1 it is at the same site with a particle, call it B, that had the opposite label in η_0. Notice that in this case the labels of A and B are on successive positions. Thus, labels can only disappear in pairs, with the "annihilation" of two successive labels of opposite types. Moreover, when a particle looses its label it does not change lanes.

(2) A particle, call it C, that is isolated in η_0 and unable to move will no longer be isolated in η_1 only if it is now at the same site with a particle, call it D, that moved into that site.

(2a) If D was isolated then this situation has been analyzed in (1) with A and B playing the roles of D and B respectively. In this case, we have an annihilation of two successive labels and none of the considered particles changes lanes.

(2b) If, on the contrary, D was not isolated in η_0 then its companion in η_0, call it E, had no label in η_0 but "inherits" the label of C in the original isolated particle in η_0. Thus, in this situation we observe a label moving backwards by 1 and none of the considered particles changes lanes.

(3) A particle, call it A, that is not isolated in η_0 will be isolated in η_1, and therefore labeled, in two situations.

(3a) It moves while its companion in η_0, call it B, does not move and the particle, call it C, that blocked the particle B, moves as well. Notice that in this situation C must be labeled in η_0. Suppose its label is "up"; the opposite case admits similar analysis. Since $\eta_0 \in \mathscr{S}$, the nearest label to the left of A and B in η_0 is "down". Thus, the labels of A and B appear in the correct order in between two already existing labels. Certainly, any of them may be annihilated in η_1. But these potential annihilation events do not change the alternating order of labels, as we have shown in (1) and (2). Notice that neither A nor B change lanes in this situation.

(3b) The second possibility is that A acquires a label because it does not move and its companion in η_0, call it B, moves. This situation becomes identical to the one considered in (3a), if one interchanges the names of A and B. Thus, this situation neither leads to the break up of the alternating order of labels nor to lane changes.

In the second step of the proof we construct an auxiliary 3-state cellular automaton in $\mathbf{Z}$ that only keeps track of how many cars there are on each position[6]. In this mapping to each position i in $\mathscr{G}$ is assigned a site

[6] This automaton is a special case of the Burgers CA introduced in [NT1].

i in $\{0, 1, 2\}^{\mathbf{Z}}$. If η_n, $n \in \mathbf{N}$, is a TL184 configuration we use M from (5) to define $\xi_n \in \{0, 1, 2\}^{\mathbf{Z}}$ as

$$\xi_n := M(\eta_n).$$

The TL184 evolution rule induces a cellular automaton on the 3-state configuration space which may be defined by an operator S on $\{0, 1, 2\}^{\mathbf{Z}}$ with $\xi_{n+1} = S(\xi_n)$ as follows

$$S(\xi_n)(i) \equiv \xi_n(i) - \min\{\xi_n(i), 2 - \xi_n(i+1)\} + \min\{\xi_n(i-1), 2 - \xi_n(i)\}. \tag{16}$$

This follows once we have verified that

$$S(M(\eta_n)) = M(T(\eta_n)) \tag{17}$$

either by inspection of the relation of TL184 to the 3-state cellular automaton presented in Table 1 or directly using the definitions of S, M given above and the explicit expression for T given by:

$$\begin{aligned} T(\eta)(i, j) = {} & \eta(i, j)\eta(i+1, j)[\eta(i+1, j) + \eta(i+1, j') - \eta(i+1, j)\eta(i+1, j')] \\ & (1 - \eta(i-1, j))(1 - \eta(i, j))\eta(i-1, j')\eta(i, j') \\ & \eta(i-1, j)(1 - \eta(i, j)) \end{aligned} \tag{18}$$

for all $(i, j) \in G$, with $j \neq j'$.

We note that no information pertaining to the particle current in TL184 is lost in its mapping into the 3-state cellular automaton, because of the relation

$$N^n_{(0,1)}(\eta) = \min\{M(\eta_n(0)), 2 - M(\eta_n(1))\} \tag{19}$$

where $N^n_{(0,1)}$, defined at the beginning of this section, denotes the number of particles that hop from position 0 to position 1 in G at time n in TL184, starting from η.

In the final step of the proof we construct a measure m supported by the set $\{(\eta, \zeta) : \zeta = L^1(\eta)) \text{ or } \zeta = L^2(\eta)\} \subset \mathscr{G} \times \mathscr{S}$ (recall that L^1 and L^2 have been defined in the beginning of this section) by setting

$$m\{(\eta, L^1(\eta)) : \eta \in A\} = m\{(\eta, L^2(\eta)) : \eta \in A\} := \tfrac{1}{2}\mu(A), \qquad \forall \text{ cylinder } A \subseteq \mathscr{G} \tag{20}$$

For any pair (η, ζ) from the support of m, it is true that $N^n_{(0,1)}(\eta) = N^n_{(0,1)}(\zeta)$, since $M(\eta) = M(\zeta)$ and because of (19). Notice also that, by our construction (20), the first (resp., second) marginal of m is μ (resp., ν from (6)). Thus,

$$J^{\mathrm{TL184}}_{t=n}(\mu)=J^{\mathrm{TL184}}_{t=n}(\nu), \qquad \forall n. \tag{21}$$

But we have shown that the particles in TL184, starting from a configuration from $\mathscr{S}$, never change lanes. Thus,

$$J^{\mathrm{TL184}}_{t=n}(\nu)=J^{\mathrm{DCA184}}_{t=n}(\nu)=J^{\mathrm{CA184}}_{t=n}(\nu_1)+J^{\mathrm{CA184}}_{t=n}(\nu_2), \qquad \forall n \tag{22}$$

The relation (12) follows from (21) and (22), since $J^{\mathrm{CA184}}_{t=\infty}(\nu_i)$, $i=1, 2$, exists, as it has been proved in [BF2] (and reviewed at the beginning of this section and in Note added 2). The actual values of $J^{\mathrm{CA184}}_{t=\infty}(\nu_i)$, $i=1, 2$, may be calculated following (7)–(8)–(9)–(10). Plugging them in (12) provides the value of $J^{\mathrm{TL184}}_{t=\infty}(\mu)$. ∎

3. THE LANE ASYMMETRY AND RELATED PHENOMENA

As a consequence of (12) (established in Theorem 1) and of the fact that $\rho(\nu_1)=\rho(\nu_2)$ (which follows from (6) and the translation invariance of μ), we can state that with respect to the total current the TL184 model is equivalent to two CA184 with equipartition of particles among both lanes. In this section we investigate the long-time limit behavior of TL184 and show (Theorem 2) that this equipartition in fact does not occur from natural asymmetric translationally invariant initial measures like product measures and that stationary velocities and currents may differ in both lanes. Of course there is no contradiction here. The point is that the value of the current does not give much information about how the particles are distributed among the two lanes.

Considering velocity and current for each individual lane makes sense since we show (Proposition 1) that from an initial product measure, with probability one each particle changes lanes only a finite number of times and therefore its evolution eventually settles down to that of the one-lane version. Natural candidates for invariant measures for TL184 are those on which particles are placed in each lane according to an invariant measure for CA184. But, as we verify in Proposition 2, these measures are not always invariant. In Theorem 3 we discuss the evolution of TL184 starting from this kind of non-invariant measures in the case where only one lane is initially occupied and find (Corollary 3) an interesting lack of monotonicity.

We consider basically three classes of measures as initial measures for TL184. The simplest translationally invariant initial measures are product measures with different densities on both lanes. Denote by μ_{ρ_1,ρ_2} the product measure with density $0 \leqslant \rho_1 \leqslant 1$ on the upper lane and $0 \leqslant \rho_2 \leqslant 1$ on the lower lane and by $\mathscr{M}$ the class of all those measures.

Another interesting class of measures will be denoted by $\mathscr{M}^{184}$. Its generic element, denoted by $v_{\rho_1}^{184} \times v_{\rho_2}^{184}$, places particles on the upper and lower lane according to independent measures, $v_{\rho_1}^{184}$, with density ρ_1, and $v_{\rho_2}^{184}$, with density ρ_2, respectively, where $v_\rho^{184} \in \mathscr{I}^{184}$ denotes the invariant measure for CA184, starting from the Bernoulli measure with the density ρ. The structure of v_ρ^{184} has been analyzed [BF2] (see Note added 2 at the end). The result is that it is translation invariant, and if $\rho \geqslant 1/2$ then it does not allow any two neighboring sites be vacant, i.e., v_ρ^{184} is supported by configurations consisting of blocks of particles separated by single vacant sites. The Bernoulli structure of the initial distribution then guarantees that the density of blocks with an even number of particles is strictly between 0 and 1, in v_ρ^{184}. This particular property is essential for the proof of Theorem 3. If $\rho \leqslant 1/2$ then v_ρ^{184} looks like $v_{1-\rho}^{184}$ in which particles and vacancies are interchanged.

Finally, another interesting class of measures consist of those obtained by taking the limit $p \to 1$ in the stationary measures of the stochastic variant of CA184 where particle hop with probability p if they can ([Y], [SSNI]). We shall denote this class of measures by $\mathscr{I}_1^{184}$.

To investigate the effect of lane changes on the stationary densities on both lanes for TL184 we first consider the case where the initial measure is $\mu_{\rho,0}$, that is, no particle is on lane 2 and particles on lane 1 are placed according to a Bernoulli measure with density ρ.

When η_0 from TL184 is chosen according to a translationally invariant measure μ then the quantities

$$P_\mu[\eta_n(i,j)=1, \eta_n(i+1,j)=0], \qquad P_\mu[\eta_n(i+1,j)=0 \mid \eta_n(i,j)=1] \tag{23}$$

do not depend on $i \in \mathbf{Z}$, and are called *the current of particles* and *the particle velocity* on the lane j at time n. Their *long time limits* are defined as $\lim_{n\to\infty}$ of the respective quantities and denoted, respectively, by $J_\mu(j)$ and $v_\mu(j)$. That these quantities actually exist and make physical sense is the content of

Proposition 1. Let the initial measure for the TL184 be either $\mu_{\rho,0}$, or $\mu_{\rho,1}$, or $\mu_{0,\rho}$, or $\mu_{1,\rho}$ from $\mathscr{M}$ for some $\rho \in [0, 1]$. Then, with probability one, each particle changes lane finitely many times.

We postpone the proof of Proposition 1 because it rests on the technique that we shall develop to prove the following theorem. It shows that the stationary densities, velocities and currents may be different on both lanes.

Theorem 2 (Lane asymmetry from Bernoulli measures). Let $\mu_{\rho,0} \in \mathcal{M}$, for some $\rho \in (0, 1)$, be the initial measure for TL184. Then

a) $\rho^+ := \lim_{n\to\infty} \mathbf{E}_{\mu_{\rho,0}} \eta_n(i, 1) \neq \lim_{n\to\infty} \mathbf{E}_{\mu_{\rho,0}} \eta_n(i, 2) =: \rho^-$
b) $v_{\mu_{\rho,0}}(1) = v_{\mu_{\rho,0}}(2) = 1$
c) $J_{\mu_{\rho,0}}(1) = \rho^+$ and $J_{\mu_{\rho,0}}(2) = \rho^-$

Remark 1. The actual values for these limit densities are (Figure 1)

$$\rho^+ = \frac{\rho}{1+\rho} \qquad \text{and} \qquad \rho^- = \frac{\rho^2}{1+\rho}.$$

Remark 2. By the lane exchange and particle-vacancy symmetries (2) the corresponding results for $\mu_{0,\rho}$, $\mu_{\rho,1}$ and $\mu_{1,\rho}$ follow. In particular, for $\mu_{\rho,1}$ the stationary velocities in both lanes will be different.

Proof of Theorem 2. Let us call a configuration η a *free* configuration if all particles are able to move. This property is preserved by the dynamics of TL184: if the configuration of the TL184 at time n, η_n, is free then η_{n+1} will also be free. Indeed, let $\tilde{\eta}(i) = \eta(i, 1) + \eta(i+1, 1) +$

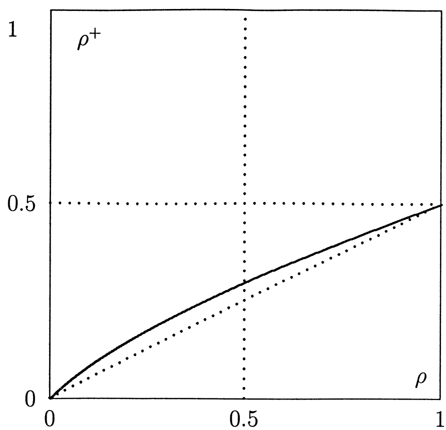

Fig. 1. The x-axis corresponds to ρ, the density of particles on lane one in the measure $\mu_{\rho,0} \in \mathcal{M}$. We show ρ^+, the long time limit of the particle density on lane one in TL184 that starts from $\mu_{\rho,0}$ (full curve) and the value of ρ^+ one would obtain if an equipartition of particles would take place (dotted curve).

$\eta(i,2)+\eta(i+1,2)$ be the total number of particles at the four sites on positions i and $i+1$ in configuration η. It is straightforward to check that a configuration η is free if and only if $\tilde{\eta}(i) \leqslant 2$ for all $i \in \mathbf{N}$. Using this and noticing that $\tilde{\eta}_n(i-1)=\tilde{\eta}_{n+1}(i)$ if η_n is free it follows that the dynamics preserves this property.

All configurations in the support of $\mu_{\rho,0}$ have particles only on lane 1, and therefore are free. On this lane one has blocks of nearest neighbor particles separated by empty sites. As the configuration is free, the evolution of the particles in separated blocks are independent and the evolution of each block is very simple to describe. Blocks with an even number of particles, say with $2k$ particles, for some $k \in \mathbf{Z}$, evolve to a configuration with k particles in each lane, occupying alternating positions, with the rightmost particle in lane 1, and therefore are equally distributed among both lanes (see Figure 2). Blocks with an odd number of particles, say $2k+1$, on the other hand, will evolve to a configuration with $k+1$ particles on alternating positions on lane 1 and only k particles on the other lane. Therefore each initial odd-sized block of particles leaves an extra particle on lane 1, namely the left-most particle of this initial block. Since the probability that a block is odd-sized is positive, the theorem is proven. The results for the stationary velocity and current immediately follow from the structure of the limit measure determined by the block dispersion mechanism that has been described above. ▌

To verify the explicit expression given in Remark 1 we note that by translational invariance the density of "extra particles" on lane 2 is equal to the probability that a given position, the origin say, is (initially) occupied by the left-most particle belonging to an odd-sized block. If we call this probability α, a simple computation using the product form of the initial measure gives

$$\alpha=\frac{\rho(1-\rho)}{1+\rho}.$$

The final densities ($\rho^{\pm}$) on lane 1 and 2 resp. are then given by

$$\rho^{-}=\frac{\rho-\alpha}{2} \qquad \text{and} \qquad \rho^{+}=\frac{\rho+\alpha}{2}$$

which yields the result presented in the remark.

	○ ○ ○ ○ ○○		○ ● ● ● ○○		○ ○ ○ ● ○○		○ ● ○ ● ○○	
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Fig. 2. The figure presents the evolution of a block of size 4 under the TL184 dynamics.

We now prove Proposition 1.

Proof of Proposition 1. When the initial measure is $\mu_{\rho,0}$ (or $\mu_{0,\rho}$), the statement follows directly from the block dispersion mechanism as described in the proof of Theorem 2.

Suppose that the initial measure is $\mu_{\rho,1}$. Since the lower lane is full of particles all the vacancies on the upper lane will be free to move (backwards) and, by the same argument used in the proof of Theorem 1, this property is preserved under the evolution. A tagged particle can only change lanes if a vacancy also does. But a block of $k \geqslant 2$ vacancies, initially on lane 1 needs $k-1$ time steps to evolve into the configuration with vacancies on both lanes, on alternating positions. Reaching this "final" configuration the vacancies will not change lanes anymore. Now, chose a particle, (the first at the right of the origin, say), and let A_n be the event that there is a block of vacancies $n \geqslant 0$ positions to its right so large that it cannot evolve into the final alternating vacancy configuration before its leftmost vacancy reaches this particle. It is easy to check that $\sum P(A_n) < \infty$, and a simple Borel Cantelli argument shows that, with probability 1, only a finite number of those large vacancy-blocks will be at the right of the origin and hence the tagged particle changes lanes only a finite number of times. By lane-exchange symmetry the same statement follows for initial measure $\mu_{1,\rho}$. ∎

It is obvious that all measures $\nu_{\rho_1}^{184} \times \nu_{\rho_2}^{184}$ are invariant if the densities are either both larger than 1/2 or both smaller than 1/2. This is clear (from the description of ν_ρ given in the beginning of Section 3) once we note that in ν_ρ^{184}, with $0 < \rho < 1/2$, all particles are free to move and therefore move with mean velocity 1. If each lane starts according to one of these measures their evolutions would be independent. The argument for the case when both densities are larger than 1/2 is analogous by considering the (backward) movement of vacancies. Hence such measures do not constitute interesting initial measures for the investigation of lane-change phenomena.

On the other hand, we have the following result:

Proposition 2. If either $\rho_1 < 1/2 < \rho_2$ or $\rho_2 < 1/2 < \rho_1$, then $\nu_{\rho_1}^{184} \times \nu_{\rho_2}^{184} \in \mathscr{M}^{184}$ is not invariant for TL184.

Proof. The idea of the proof is as follows. We first map the evolution of TL184, from $\nu_{\rho_1}^{184} \times \nu_{\rho_2}^{184}$, to that of an annihilating-particle system. In this system particles moving in both directions in $\mathbf{Z}$ are initially placed according to some translationally invariant measure. In this dynamics there is no creation of annihilating-particles (a.p., for short) and when two a.p.

moving in opposite directions meet, they annihilate each other. For $\rho_1 < 1/2 < \rho_2$ or $\rho_2 < 1/2 < \rho_1$ we verify that the density of a.p. decreases with time and therefore $\nu^{184}_{\rho_1} \times \nu^{184}_{\rho_2}$ is not invariant.

Each site $i \in \mathbf{Z}$ of the annihilating-particle system may be empty, occupied by a single a.p. moving either to the left or to the right or occupied by two a.p. both moving to the left or to the right. Let us denote by α a configuration of this annihilating-particle system with $\alpha(i) \in \{-2, -1, 0, 1, 2\}$ so that a site with state $\alpha(i)$ indicates the presence there of $|\alpha(i)|$ annihilating particles moving to the left if $\alpha(i)$ is negative or moving to the right otherwise. The dynamics of this annihilating system proceeds in two steps. At each time step: first each a.p. moves one position in its predefined direction of movement; then any two particles moving in opposite directions that find themselves in the same site, annihilate each other. Note that since the distance of two a.p. moving in opposite directions can only increase or decrease by 2 in each time step, annihilating particles on the even and on the odd sub-lattices of $\mathbf{Z}$ do not interact with each other. The mapping of $\mathscr{G}$ into the annihilating-particle system is defined as follows: if η_n is the state of the TL184 cellular automaton at time n then the corresponding state, α_n, of the annihilating-particle system is given by $\alpha_n(i) = 2 - \tilde{\eta}_n(i)$, $i \in \mathbf{Z}$, where $\tilde{\eta}_n(i) = \eta_n(i, 1) + \eta_n(i+1, 1) + \eta_n(i, 2) + \eta_n(i+1, 2)$, is the total number of TL184-particles at the four sites on positions i and $i+1$ in configuration η_n, as in the proof of Theorem 2. Indicating by $\ll$, $<$, $*$, $>$ and $\gg$ the states -2, -1, 0, 1 and 2, respectively, the graphical representation (24) indicates part of a configuration for the two lane system and the corresponding configuration of annihilating particles

$$\begin{array}{cccccccccccccc} & \bullet & \circ & \circ & \bullet & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \circ & \circ & \\ \cdots & * & > & * & < & > & * & \ll & < & * & * & < & ? & \cdots \\ & \circ & \bullet & \circ & \bullet & \circ & \circ & \bullet & \bullet & \bullet & \circ & \bullet & \circ & \end{array} \tag{24}$$

The state of the a.p. system corresponding to the last position to the right can not be determined with the information available in the figure and therefore is indicated by a question mark.

It is not difficult to find the evolution rules for the a.p. induced by TL184 evolution rules. For $i \in \mathbf{Z}$ and $n \geqslant 1$ we find $\alpha_{n+1}(i) = \Psi(\alpha_n(i-1), \alpha_n(i+1))$, where $\Psi(a, b) = a - [a]_- + [b]_-$ and $[a]_- = |x|$ if $x < 0$, and $[a]_- = 0$ otherwise.

Suppose $\rho_2 < 1/2 < \rho_1$. Under $\nu^{184}_{\rho_1} \times \nu^{184}_{\rho_2}$ from $\mathscr{M}^{184}$ the upper lane (lane 1) has blocks of successive sites occupied by particles separated by blocks of alternating occupied and empty sites while on the lower lane there are blocks of successive empty sites that are separated by blocks of alternating occupied and empty sites. Since $\nu_{\rho_1}^{184}$ and $\nu_{\rho_2}^{184}$ are independent the corresponding annihilating particle system has, on both sub-lattices of $\mathbf{Z}$, positive density of particles moving in both directions. Since the distribution of a.p. is translationally invariant and annihilation necessarily decreases the density of those particles during the evolution, the proposition follows. ∎

From Proposition 2 it is natural to ask what happens if a non-invariant measure $\nu_{\rho_1}^{184} \times \nu_{\rho_2}^{184}$ is taken as the initial measure. Starting with this kind of measure corresponds to a two-step process where one first lets two independent CA184 reach equilibrium from some initial state and only then allows for lane change, i.e., one starts with DCA184 rules and then connects the two lanes to evolve according to the dynamics of TL184. As remarked before, a particularly interesting situation is when the original measure was Bernoulli on each lane with two different densities. In this case the first step of the evolution leads to a measure $\nu_{\rho_1}^{184} \times \nu_{\rho_2}^{184} \in \mathscr{M}^{184}$ which then plays the role of the initial measure for the second step, i.e., evolution under TL184. For an initial measure $\nu_{\rho_1}^{184} \times \nu_{\rho_2}^{184} \in \mathscr{M}^{184}$ with $\rho_1 = 0$ and $\rho_2 > 1/2$ we now show that both stationary densities are smaller than 1/2. This is somewhat contrary to intuition as one might believe that the loss of particles should stop once a stationary state with density 1/2 is reached on lane 2 (and hence the system as a whole would be stationary).

Theorem 3. Let $\nu_0^{184} \times \nu_\rho^{184} \in \mathscr{M}^{184}$ be the initial measure for TL184. If ρ is neither 1/2 nor 1, then

$$\varrho^- \equiv \lim_{n\to\infty} \mathbf{E}_{\nu_0^{184}\times\nu_\rho^{184}}[\eta_n(i,1)] < \tfrac{1}{2} \qquad \text{and} \qquad \varrho^+ \equiv \lim_{n\to\infty} \mathbf{E}_{\nu_0^{184}\times\nu_\rho^{184}}[\eta_n(i,2)] < \tfrac{1}{2}.$$

This result also implies an interesting non-monotonicity formulated in Corollary 3 and illustrated by Fig. 3. Since (trivially) the stationary density ϱ^+ on lane two is ρ for $\rho \leqslant 1/2$ and since also $\varrho^+ = 1/2$ for $\rho = 1$, but strictly less than 1/2 for $\rho \in (1/2, 1)$, the limiting density ϱ^+ is a non-monotonic function of ρ.

Corollary 3. Let, as in Theorem 3, $\nu_0^{184} \times \nu_\rho^{184} \in \mathscr{M}^{184}$ be the initial measure for TL184. Then the stationary density in the lane that starts occupied is not monotonic with respect with the initial density.

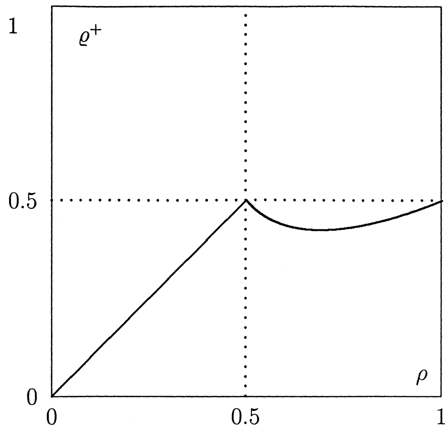

Fig. 3. The x-axis corresponds to ρ, the density of particles on lane two in the measure $\nu_0 \times \nu_\rho$ with $\nu_\rho \in \mathscr{I}_1^{184}$. The y-axis corresponds to ϱ^+, the long time limit of the particle density on lane two in TL184 that starts from $\nu_0 \times \nu_\rho$. Qualitatively, the same relation of ϱ^+ to ρ holds for $\nu_0^{184} \times \nu_\rho^{184} \in \mathscr{M}^{184}$.

The actual values of the limit densities are not difficult to compute in some special cases. An interesting case is when the initial measure is $\nu_0 \times \nu_\rho$ with $\nu_\rho \in \mathscr{I}_1^{184}$ where straightforward calculation yields limiting densities

$$\varrho^+ = \frac{\rho^2}{3\rho - 1} \qquad \text{and} \qquad \varrho^- = \frac{\rho(2\rho - 1)}{3\rho - 1}.$$

for $\rho \geqslant 1/2$.

Proof of Theorem 3. In order to prove this result we first have to describe the measure $\nu_0^{184} \times \nu_\rho^{184}$. We only need to describe it on lane 2 where it is a measure on $\{0, 1\}^{\mathbb{Z}}$ corresponding to the limit measure for CA184 starting from the Bernoulli measure with density $\rho > 1/2$. This measure does not allow for pairs of neighboring vacancies. The allowed configurations consist of blocks of particles separated by blocks of alternating particles and vacancies. Thus in the initial measure for TL184 all particles are free and remain free during the evolution. Each block of nearest neighbor particles, initially in lane 2, will evolve, independently from each other, so that the rightmost particle remains on lane 2. The particle initially to its left goes to lane 1 and so on, forming an alternating particle/vacancy block. Consider one of those initial blocks of particles with length k. To its left and to its right there are blocks of alternating particles/vacancies. If k is odd the leftmost particle of this k-sized block ends up in lane 2 and a larger

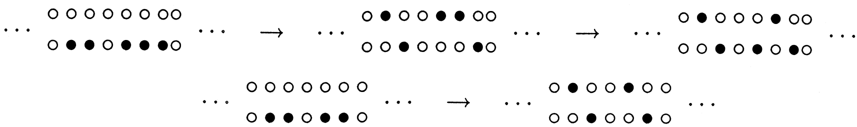

Fig. 4. Particles are marked by $\bullet$, vacancies are marked by $\circ$. The upper (resp., lower) line shows the creation of a pair of consecutive particles separated by one (resp., two) vacant sites. In both cases, the leftmost particle of the pair is the first particle of the leftmost block in the initial configuration.

alternating particle/vacancy block is formed on this lane (as illustrated by the upper line of Fig. 4). On the other hand, if k is even there will be a "gap" of two vacancies separating two alternating particle/vacancy blocks (as illustrated by the lower line of Fig. 4).

Therefore the presence of even sized particle blocks in lane 2 in the initial configuration leads to "extra vacancies" there, which result in a stationary density smaller than 1/2. It is simple to verify that those even sized blocks occur with positive probability and therefore the theorem is proven. ▮

4. REMARKS ON A CONTINUOUS TIME VERSION OF TL184

In the framework of a continuous-time description of two-lane traffic flow one may describe the state of the system in terms of road-segment occupation numbers $\xi(i) = 0, 1, 2$ in a similar fashion as when passing from TL184 to the restricted three-state process. Developing further the ideas set out in the introduction we define a model on the basis of the following first principles

- (A) exclusion in each lane.
- (B) totally asymmetric hopping with exponential waiting-time distribution.
- (C) in each lane all cars behave identically, but there is a slow and a fast lane.
- (D) cars move to the lower lane (lane 2) whenever possible.

Rule (D) is motivated by traffic regulations in Germany and other countries which allow cars to use one lane (the "fast" lane) only for passing as long as the slow lane can still accommodate vehicles. On a technical level, the asymmetric lane-changing rule (D) is an ingredient which allows us to uniquely identify a state of the three-state exclusion process with a configuration on the two-lane road, viz. state '1' refers to a car on the lower lane 2. We remark that an effective one-lane description of traffic flow with passing is implicit also in other recent work ([IK1], [IK2]).

Notice that (C) is a generalization of our previous considerations where cars moved on both lanes in an identical manner. On lane 2 each car moves with constant rate α. On lane 1, which we define as passing lane, cars move with rate γ. This results in the following dynamics: (i) If a car is on site k of lane 2, and the next site $k+1$ on lane 2 is empty, the car moves onto this site with rate α. (ii) If the next site on lane 2 is occupied, but the site on the passing lane 1 is vacant, the car on site k on lane 2 will pass, i.e. will move to site $k+1$ on lane 1. This transition $11 \to 02$ occurs with rate β which is a free parameter. (iii) If there are cars on site k on both lanes, and there is also a car on site $k+1$ of lane 2, then the car on lane 1 cannot move back to lane 2. Instead it will hop onto site $k+1$ of lane 1 with rate γ which fixes the speed on the passing lane. (iv) Finally, when there are two cars on site k, but site $k+1$ is empty on both lanes, then one of two things can happen. Either the car on lane 2 proceeds (with rate α) and at the same time the car on lane 1 moves to lane 2 (but remaining at site k), or the car on lane 1 proceeds first onto site $k+1$ of lane 2 (with rate γ). In terms of occupation numbers, both transitions are indistinguishable, hence $\delta = \alpha + \gamma$ for the transition $20 \to 11$. Thus the model is defined by the transitions

$$\begin{aligned} &10 \to 01 \text{ with rate } \alpha \\ &11 \to 02 \text{ with rate } \beta \\ &21 \to 12 \text{ with rate } \gamma \\ &20 \to 11 \text{ with rate } \delta = \alpha + \gamma. \end{aligned} \tag{25}$$

The constraint on δ resulting from the definition that cars on lane 2 (1) move with rate α (γ) has an intriguing consequence. It is straightforward to show that the invariant measures contain a family of translationally invariant product measures μ where the stationary probabilities p_1 of finding one car on site k and p_2 of finding both lanes occupied satisfy the relation

$$\beta p_1^2 = (\alpha + \gamma)\, p_2 (1 - p_1 - p_2). \tag{26}$$

We note that the individual lane densities are given by $\rho_1 = p_1 + p_2$ and $\rho_2 = p_2$ respectively. Hence

$$\rho = p_1 + 2 p_2. \tag{27}$$

ranging from 0 to 2 is the conserved total density which parameterizes the family of measures μ.

The fundamental quantity of interest is the stationary current $j=\alpha\langle 10\rangle+\beta\langle 11\rangle+\gamma\langle 21\rangle+\delta\langle 20\rangle$ where

$$\langle mn\rangle=\mathbf{E}_{\mu}(\xi(i)=m,\ \xi(i+1)=n) \tag{28}$$

for the stationary product measure μ with density ρ. In what follows we set $\alpha=1$ which is a normalization of the time scale of the process and involves no loss of generality. Since in the event of passing a car starts from lane 2 where cars move with rate α, we shall further assume $\beta=\alpha$. Thus $j=p_1(1-2p_2)+(1+\gamma)\,p_2(1-p_2)$ where we use $\delta=1+\gamma$. With ref. 26, ref. 27 one can express j as a function of ρ to obtain the flow diagram of the two-lane traffic model. Straightforward calculation yields

$$j=\rho-\rho^2+\left(\gamma+\frac{1-\gamma}{2}\right)\left[\frac{1+\gamma}{3-\gamma}+\rho-\sqrt{\left(\frac{1+\gamma}{3-\gamma}+\rho\right)^2-\frac{4\rho^2}{3-\gamma}}\right]. \tag{29}$$

The worst realistic case corresponds to $\gamma=\alpha=1$ where cars in the fast lane move with the same average speed as in the slow lane. The flow-density relation becomes

$$j=1+2\rho-\rho^2-\sqrt{1+2\rho-\rho^2}. \tag{30}$$

For *all* densities ρ this current is larger than the maximal current attainable in two single non-interacting lanes at the same total density. Numerical studies of a similar two-lane traffic model with the same transitions as those shown in (25), but without the constraint $\delta=\alpha+\gamma$, also yield current-enhancement [Fo].

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Note Added 1. After the original submission of our paper we became aware of Ref. [NT2] where TL184 is also suggested as a traffic flow model.

Note Added 2. Since we use results from [BF2] that has not yet appeared in print, we shall indicate here the way these results have been derived.

In [BF2], we consider a discrete-time process called *Ballistic Annihilation* (abbreviated by BA). Its state space is $\{-1, 0, +1\}^{\mathbf{Z}}$. For arbitrary $\zeta \in \{-1, 0, +1\}^{\mathbf{Z}}$ and $x \in \mathbf{Z}$, let us interpret the values $0, +1, -1$ of $\zeta(x)$ by saying that the site x is respectively, free of an A-particle, contains an A-particle with velocity $+1$, and contains an A-particle with velocity -1 ("A" stands for "annihilating", in order to distinguish A-particles from the particles in CA184). In terms of A-particles, the dynamics of BA is defined as follows: each A-particle moves along $\mathbf{Z}$ with its velocity, going in the direction of $-\infty$ $(+\infty)$, if the velocity is negative (positive, resp.). When meeting another A-particle both A-particles annihilate, i.e., they disappear from the system forever (note that the meeting point may not be an integer).

Let $\mathbf{P}_-^{\mathrm{BA}}$ (resp., $\mathbf{P}_+^{\mathrm{BA}}$) denote the set of the extremal measures of the set of the translation invariant measures on $\{-1, 0\}^{\mathbf{Z}}$ (resp., on $\{0, +1\}^{\mathbf{Z}}$). We show in [BF2] that a translation invariant measure μ on $\{-1, 0, +1\}^{\mathbf{Z}}$ is invariant for BA if and only if it belongs to the convex hull of the set $\mathbf{P}_-^{\mathrm{BA}} \cup \mathbf{P}_+^{\mathrm{BA}}$. The "if" part is easy since it may be checked straightforwardly that any element from the latter set is invariant for BA. The "only if" part follows from the fact that a translation invariant measure μ which is invariant for BA, gives weight zero to any configuration $\zeta \in \{-1, 0, +1\}^{\mathbf{Z}}$ that contains both A-particles with positive and A-particles with negative velocity; it is not straightforward, but follows from a simple analysis of the transformation of such ζ by the dynamics of BA. We also show in [BF2] that starting from any translation invariant measure μ, BA converges to some μ^∞ from the convex hull of $\mathbf{P}_-^{\mathrm{BA}} \cup \mathbf{P}_+^{\mathrm{BA}}$. This happens because only A-particles with the same velocity may survive forever in BA that starts from a configuration ζ from the support of such μ (certainly, the velocity of the surviving A-particles may depend on ζ).

Let us introduce the mapping $T_{184,\,\mathrm{BA}}: \{0, 1\}^{\mathbf{Z}} \to \{-1, 0, +1\}^{\mathbf{Z}}$ by

$$(T_{184,\,\mathrm{BA}}\eta)(i) = 1 - \eta(i) - \eta(i+1), \qquad i \in \mathbf{Z} \tag{31}$$

It may be verified straightforwardly that if a process $\{\eta_n\}_{n \in \mathrm{N}}$ is CA184 then the process $\{T_{184,\,\mathrm{BA}}\eta_n\}_{n \in \mathrm{N}}$ is BA ([BF1],[KS]). This relation and the properties of BA from the above paragraph imply that CA184, starting from any translation invariant measure λ, converges to some translation invariant measure λ^∞ that may be represented as

$$\lambda^\infty = \sum_i \alpha_i \lambda_i^\infty, \text{ where for each } i,\ \lambda_i^\infty = T^*_{184,\,\mathrm{BA}}(\mu_i) \text{ for some } \mu_i \in \mathbf{P}_-^{\mathrm{BA}} \cup \mathbf{P}_+^{\mathrm{BA}} \tag{32}$$

Above, $T^*_{184,\,\mathrm{BA}}$ maps a particular subset of the measures on $\{-1, 0, +1\}^{\mathbf{Z}}$ to the set of the measures on $\{0, 1\}^{\mathbf{Z}}$; it is naturally induced by $T_{184,\,\mathrm{BA}}$. Although $T_{184,\,\mathrm{BA}}$ is not a bijection, $T^*_{184,\,\mathrm{BA}}$ is thoroughly characterized in [BF2]. Note that the structure of v^{184}_{ρ} described and employed in Section 3, follows from the convergence just stated, since $v^{184}_{\rho} \equiv \lambda^{\infty}$ for a Bernoulli measure λ with the particle density equal to ρ.

We now note that if $\lambda^{\infty} = T^*_{184,\,\mathrm{BA}}(\mu)$ for $\mu \in \mathbf{P}^{\mathrm{BA}}_{+}$ then, due to (31), no two particles in λ^{∞} may occupy neighboring sites of $\mathbf{Z}$, and thus, each particle will be always able to move in CA184, starting from λ^{∞}. Thus, $J^{\mathrm{CA184}}(\lambda^{\infty}) = \rho(\lambda^{\infty})$. A similar reasoning implies that $J^{\mathrm{CA184}}(\lambda^{\infty}) = 1 - \rho(\lambda^{\infty})$, when $\mu \in \mathbf{P}^{\mathrm{BA}}_{-}$.

Suppose now that a translation invariant measure λ satisfies (8). Let $\mu := (T^*_{184,\,\mathrm{BA}})^{-1}(\lambda)$ (it may be shown that μ is well defined). It then follows from (8) and (31) that for any A-particle with negative velocity in μ, there exists, μ-almost surely, an A-particle with positive velocity that will annihilate it sooner or later in BA, starting from μ. Using the relation of CA184 to BA induced by (31), we conclude then, that there is no A-particle with negative velocity in any μ_i from the expansion (32) for λ^{∞}, the asymptotic measure of CA184, starting from λ. Thus, $J^{\mathrm{CA184}}_{t=\infty}(\lambda) \equiv J^{\mathrm{CA184}}(\lambda^{\infty}) = \sum_i \alpha_i \rho(\lambda^{\infty}_i) = \rho(\lambda^{\infty})$. But since CA184 does not change the particle density of λ, then $J^{\mathrm{CA184}}_{t=\infty}(\lambda) = \rho(\lambda)$. A similar consideration gives that $J^{\mathrm{CA184}}_{t=\infty}(\lambda) = 1 - \rho(\lambda)$, when λ satisfies (9). The general property expressed by (7), (8), (9), (10) follows by the same consideration, when we "split" λ in two parts, one that satisfies (8), and the other that satisfies (9).

REFERENCES

[BF1] V. Belitsky and P. A. Ferrari, Ballistic annihilation and deterministic surface growth, *J. Stat. Phys.* **80**(3/4) (1995).

[BF2] V. Belitsky and P. Ferrari, Invariant measures and convergence for cellular automaton 184 and related processes, submitted to *J. Stat. Phys.*

[BML] O. Biham, A. A. Middleton, and D. Levine, Self-organization and a dynamic transition in traffic flow models, *Phys. Rev. A* **46**:R6124 (1992).

[CSS] D. Chowdhury, L. Santen, and A. Schadschneider, Statistical physics of vehicular traffic and some related systems, *Phys. Rep.* **329**:199 (2000).

[CWS] D. Chowdhury, D. E. Wolf, and M. Schreckenberg, Particle hopping models for two-lane traffic with two kinds of vehicles: Effects of lane-changing rules, *Physica A* **235**:417 (1997).

[DE] J. de Gier and B. Nienhuis, Exact stationary state for an asymmetric exclusion process with fully parallel dynamics, *Phys. Rev. E* **59**:4899 (1999).

[ERS] M. R. Evans, N. R. Rajewsky, and E. R. Speer, Exact solution of a cellular automaton for traffic, *J. Stat. Phys.* **95**:45 (1999).

[Fo] M. E. Fouladvand, Reaction-diffusion models describing a two-lane traffic flow, *Phys. Rev. E* **62**:5940 (2000).

[HH] D. Helbing and B. A. Huberman, Coherent moving states in highway traffic, *Nature* **396**:738 (1998).

[IK1] I. Ispolatov and P. L. Krapivsky, Particle systems with stochastic passing, *Phys. Rev. E* **61**:2163 (2000).

[IK2] I. Ispolatov and P. L. Krapivsky, Phase transition in a traffic model with passing, *Phys. Rev. E* **62**:5935 (2000).

[KS] J. Krug and H. Spohn, Universality Classes for Deterministic Surface Growth, *Phys. Rev. A* **38**:4271 (1988).

[Na1] T. Nagatani, Self-organization and phase transition in traffic-flow model of a two-lane roadway, *J. Phys. A* **26**:L781 (1993).

[Na2] T. Nagatani, Kinetics of segregation in a two-lane highway traffic flow, *J. Phys. A* **29**:6531 (1996).

[NH] K. Nagel and H. J. Herrmann, Deterministic models for traffic jams, *Physica A* **199**:254 (1993).

[NWW] K. Nagel, P. Wagner, and D. E. Wolf, Realistic multi-lane traffic rules for cellular automata, *Physica A* **234**:687 (1997).

[NWWS] K. Nagel, D. E. Wolf, P. Wagner, and P. Simon, Two-lane traffic rules for cellular automata: A systematic approach, *Phys. Rev. E* **58**:1425 (1998).

[NT1] K. Nishinari and D. Takahashi, Analytical properties of ultradiscrete Burgers equation and rule-184 cellular automaton, *J. Phys. A* **31**:5439 (1998).

[NT2] K. Nishinari and D. Takahashi, A family of multi-value cellular automaton model for traffic flow, Preprint nlin.AO/0002007, submitted to *Phys. Rev. E*.

[RNSL] M. Rickert, K. Nagel, M. Schreckenberg, and A. Latour, Two-lane traffic simulations using cellular automata, *Physica A* **231**:534 (1996).

[SSNI] M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito, Discrete Stochastic Models for Traffic Flow, *Phys. Rev. E* **51**:2939 (1995).

[SW] M. Schreckenberg and D. E. Wolf, eds., *Traffic and Granular Flow '97* (Springer, Singapore 1998).

[TE] L. G. Tilstra and M. H. Ernst, Synchronous asymmetric exclusion processes, *J. Phys. A* **31**:5033 (1998).

[Wo] D. E. Wolf, Cellular automata for traffic simulations, *Physica A* **263**:438 (1999).

[Wm] S. Wolfram, Statistical mechanics of cellular automata, *Rev. Mod. Phys.* **55** (1983).

[Y] H. Yaguchi, Stationary measures for an exclusion process on one-dimensional lattices with infinitely many hopping sites, *Hiroshima Math. J.* **16**:449 (1986).